PART – I

1. (c)
2. (d)
3. (c)
4. (a)
5. (c)
6. (d)
7. (c)
8. (b)
9. (c)
10. (b)
11. (d)
12. (a)
13. (c)
14. (d)
15. (a)
16. (d)
17. (a)
18. (b)
19. (b)
20. (b)

PART – II

21. $\left(\frac{1+i}{1-i}\right)^3-\left(\frac{1-i}{1+i}\right)^3=-2i$

Step (1): Simplify $\frac{1+i}{1-i}$ and $\frac{1-i}{1+i}$

1(a) Simplify $\frac{1+i}{1-i}$

Multiply numerator and denominator by the conjugate of the denominator $1+i$:

$$\frac{1+i}{1-i}\cdot\frac{1+i}{1+i}=\frac{(1+i)^2}{(1-i)(1+i)}$$

- Numerator: $(1+i)^2=1+2i+i^2=1+2i-1=2i$
- Denominator: $(1-i)(1+i)=1-i^2=1-(-1)=2$

$$\frac{1+i}{1-i}=\frac{2i}{2}=i$$

So: $\frac{1+i}{1-i}=i$

1(b) Simplify $\frac{1-i}{1+i}$

Similarly, multiply numerator and denominator by conjugate $1-i$:

$$\frac{1-i}{1+i}\cdot\frac{1-i}{1-i}=\frac{(1-i)^2}{(1+i)(1-i)}$$

Numerator: $(1-i)^2=1-2i+i^2=1-2i-1=-2i$

Denominator: $(1+i)(1-i)=2$

$$\frac{1-i}{1+i}=\frac{-2i}{2}=-i$$

So: $\frac{1-i}{1+i}=-i$

Step (2): Cube both terms

$$\left(\frac{1+i}{1-i}\right)^3=i^3=-i$$

$$\left(\frac{1-i}{1+i}\right)^3=(-i)^3=-i^3=-(-i)=i$$

Step (3): Subtract

$\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = (-i) - (i) = -2i$

This proves the given expression.

22. If $(1+i)(1+2i)\cdots(1+ni) = x+iy$, then prove that $2 \cdot 5 \cdot 10 \cdots (1+n^2) = x^2+y^2$

I notice some parts are written as ___ . I will assume the meaning is:

$(1+i)(1+2i)\cdots(1+ni) = x+iy$

and we need to prove:

$$\prod_{k=1}^{n} (1+k^2) = x^2+y^2$$

Step (1): Recall the property of modulus of complex numbers

For any complex number $z = a+ib$:

$|z|^2 = a^2+b^2$

Also:

$|z_1 z_2 \cdots z_n| = |z_1| \cdot |z_2| \cdots |z_n|$

This is the key property.

Step (2): Apply the modulus property

Let:

$z = (1+i)(1+2i)\cdots(1+ni) = x+iy$

Then:

$|z|^2 = |x+iy|^2 = x^2+y^2$

Step (3): Compute the modulus of each term

$|1+ki| = \sqrt{1^2+k^2} = \sqrt{1+k^2}$

So:

$$|z| = \prod_{k=1}^{n} |1+ki| = \prod_{k=1}^{n} \sqrt{1+k^2}$$

Step (4): Square both sides

$$|z|^2 = x^2+y^2 = \left(\prod_{k=1}^{n} \sqrt{1+k^2}\right)^2 = \prod_{k=1}^{n} (1+k^2)$$

Step (5): Conclusion

$$x^2+y^2 = \prod_{k=1}^{n} (1+k^2)$$

This proves the required relation.

23. We are asked to find: -

$\sin^{-1}[\sin(5\pi/4)]$.

Let's solve this step by step.

Step (1):- Recall the range of $\sin^{-1} x$

The function $\sin^{-1} x$ (inverse sine) has the principal value range:

$\sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

This means that the answer must be within $-\pi/2$ to $\pi/2$.

Step (2): Express $5\pi/4$ in the principal range

The angle given is

$5\pi/4 = \pi + \pi/4$

But $\pi + \pi/4 = 225°$, which is outside $[-\pi/2, \pi/2]$.

Step 3: Use the sine periodic property

We know that:

$\sin(\theta) = \sin(\pi - \theta)$ or $\sin(\theta) = \sin(\theta - 2\pi k)$

for any integer k.

Also, $\sin(\pi + \alpha) = -\sin\alpha$.

Here:

$$\sin(5\pi/4) = \sin(\pi + \pi/4) = -\sin(\pi/4) = -\frac{\sqrt{2}}{2}.$$

Step (4): Find the principal value
We now want an angle θ in $[-\pi/2, \pi/2]$ such that:
$\sin(\theta) = -\frac{\sqrt{2}}{2}$.
Clearly: -
$\theta = -\pi/4$
because $\sin(-\pi/4) = -\frac{\sqrt{2}}{2}$ and $-\pi/4$ is in the interval $[-\pi/2, \pi/2]$.

24. Step (1): Formula for torque
The torque about a point $P(x_0, y_0, z_0)$ due to a force F acting along a line through position vector r is:
$\tau = (\mathrm{r} - \mathrm{r}_0) \times \mathrm{F}$
where:
- $\mathrm{r}_0 =$ position vector of the point about which torque is calculated,
- $\mathrm{r} =$ any point on the line of action of the force (here the origin, 0),
- $\mathrm{F} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$.

So here: -
$\mathrm{r} = 0, \mathrm{r}_0 = 2\hat{\imath} + 0\hat{\jmath} - 1\hat{k} = 2\hat{\imath} - \hat{k}$
$\tau = (\mathrm{r} - \mathrm{r}_0) \times \mathrm{F} = (0 - (2\hat{\imath} - \hat{k})) \times (2\hat{\imath} + \hat{\jmath} - \hat{k})$
$\tau = (-2\hat{\imath} + \hat{k}) \times (2\hat{\imath} + \hat{\jmath} - \hat{k})$
Step (2): Compute the cross product
Let
$\mathrm{A} = -2\hat{\imath} + 0\hat{\jmath} + 1\hat{k}, \mathrm{B} = 2\hat{\imath} + 1\hat{\jmath} - 1\hat{k}$
The cross-product formula is:
$$\mathrm{A} \times \mathrm{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$
Compute determinant:
(1). $\hat{\imath}$ component:
$\hat{\imath} \cdot \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = \hat{\imath}(0 \cdot (-1) - 1 \cdot 1) = -1\hat{\imath}$
(2). $\hat{\jmath}$ component (remember minus):
$-\hat{\jmath} \cdot \begin{vmatrix} -2 & 1 \\ 2 & -1 \end{vmatrix} = -\hat{\jmath}((-2)(-1) - (1)(2)) = -\hat{\jmath}(2-2) = 0\hat{\jmath}$
(3).$\hat{k}$ component: -
$\hat{k} \cdot \begin{vmatrix} -2 & 0 \\ 2 & 1 \end{vmatrix} = \hat{k}((-2)(1) - (0)(2)) = -2\hat{k}$
So: -
$\tau = -\hat{\imath} + 0\hat{\jmath} - 2\hat{k} = -\hat{\imath} - 2\hat{k}z$
Step (3): Magnitude of torque
$|\tau| = \sqrt{(-1)^2 + 0^2 + (-2)^2} = \sqrt{1+0+4} = \sqrt{5}$
Step 4: Direction cosines
Direction cosines (l, m, n) are: -
$l = \frac{\tau_x}{|\tau|}, m = \frac{\tau_y}{|\tau|}, n = \frac{\tau_z}{|\tau|}$
$l = \frac{-1}{\sqrt{5}}, m = \frac{0}{\sqrt{5}} = 0, n = \frac{-2}{\sqrt{5}}$

25. $f(x) = x + \frac{1}{x}, x \in [1/2, 2]$.
Let's solve step by step.
Step (1): Recall Rolle's Theorem
Rolle's Theorem states:
If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there exists at least one $c \in (a, b)$ such that:
$f'(c) = 0$
Step (2): Check the conditions
(1). $f(x) = x + \frac{1}{x}$ is continuous on $[1/2, 2]$ and differentiable on $(1/2, 2)$ because $x \neq 0$.

(2). Check if $f(a)=f(b)$ for $a=1/2$ and $b=2$:

$f(1/2)=\frac{1}{2}+\frac{1}{1/2}=\frac{1}{2}+2=\frac{5}{2}$

$f(2)=2+\frac{1}{2}=\frac{5}{2}$

Yes, $f(1/2)=f(2)$.

All conditions of Rolle's Theorem are satisfied.

Step (3): Find $f'(x)$ and solve $f'(c)=0$

$f(x)=x+x^{-1} \Longrightarrow f'(x)=1-\frac{1}{x^2}$

Set $f'(c)=0$:

$1-\frac{1}{c^2}=0 \Longrightarrow \frac{1}{c^2}=1 \Longrightarrow c^2=1$

$c=\pm 1$

Step (4): Choose$c \in (1/2,2)$

Only $c=1$ lies in $(1/2,2)$.

Step (5): Final Answer

$\underline{c=1}$

26. Given

$f(x)=x^2+3x$

Step 1: Find $f'(x)$

$f'(x)=2x+3$

Step (2): Use the differential formula

$df=f'(x)dx$

At $x=2$ and $dx=0.1$:

$f'(2)=2(2)+3=7$

Step (3): Calculate df

$df=7\times 0.1=0.7$

27. To prove:

$$\int_0^{\pi/2}\frac{f(\sin x)}{f(\sin x)+f(\cos x)}dx=\frac{\pi}{4}$$

Step (1): Let the given integral be

$$I=\int_0^{\pi/2}\frac{f(\sin x)}{f(\sin x)+f(\cos x)}dx$$

Step (2): Use the property $x \rightarrow \frac{\pi}{2}-x$

Make the substitution:

$x=\frac{\pi}{2}-t \;\Rightarrow\; dx=-dt$

When $x=0, t=\frac{\pi}{2}$

When $x=\frac{\pi}{2}, t=0$

Then:

$$I=\int_0^{\pi/2}\frac{f(\cos t)}{f(\cos t)+f(\sin t)}dt$$

Step (3): Add the two forms of the integral

Add the original form and the transformed form:

$$2I=\int_0^{\pi/2}\left[\frac{f(\sin x)}{f(\sin x)+f(\cos x)}+\frac{f(\cos x)}{f(\sin x)+f(\cos x)}\right]dx$$

$$2I=\int_0^{\pi/2}\frac{f(\sin x)+f(\cos x)}{f(\sin x)+f(\cos x)}dx$$

$$2I=\int_0^{\pi/2}1dx$$

Step (4): Evaluate the integral

$2I = [x]_0^{\pi/2} = \frac{\pi}{2}$

$I = \frac{\pi}{4}$

28. $y^2 = 4ax$ where a is an arbitrary constant.

Step (1): Differentiate w.r.t. x

Differentiate both sides with respect to x :

$2y\frac{dy}{dx} = 4a$

Step (2): Eliminate the arbitrary constant a

From the original equation:

$a = \frac{y^2}{4x}$

Substitute this value of a into the differentiated equation:

$2y\frac{dy}{dx} = 4\left(\frac{y^2}{4x}\right)$

Simplify:

$2y\frac{dy}{dx} = \frac{y^2}{x}$

Step (3): Simplify the differential equation

Divide both sides by y (assuming $y \neq 0$):

$2\frac{dy}{dx} = \frac{y}{x}$

or

$2x\frac{dy}{dx} = y$

29. If a binary operation on a set has an identity element, then that identity element is unique.

Proof: -

Let $(S,*)$ be a set with a binary operation $*$.

Assume that there are two identity elements in S, say e and e'.

By definition of identity:

- Since e is an identity element,

$e * a = a * e = a$ for all $a \in S$

- Since e' is also an identity element,

$e' * a = a * e' = a$ for all $a \in S$

Step (1): Operate e with e'

Using the identity property of e :

$e * e' = e'$

Using the identity property of e' :

$e * e' = e$

Step (2): Compare the results

$e = e'$

Hence, the identity element is unique if it exists.

30. $(y-k)^2 = -4a(x-h)$

where (h,k) is the vertex.

Step (1): Write the equation using the vertex

Given vertex $(2,1)$:

$(y-1)^2 = -4a(x-2)$

Step (2): Use the given point $(1,3)$

Substitute $x = 1, y = 3$:

$(3-1)^2 = -4a(1-2)$

$4 = -4a(-1)$

$4 = 4a \Rightarrow a = 1$

Step (3): Write the final equation

Substitute $a = 1$:

$(y-1)^2 = -4(x-2)$

PART – III

31. Given

$A = \begin{pmatrix} 2 & 9 \\ 1 & 7 \end{pmatrix}$

We have to prove:

$(A^T)^{-1} = (A^{-1})^T$

Step (1): Find A^{-1}

Frat compute the determinant of A:

$|A| = (2)(7) - (9)(1) = 14 - 9 = 5 \neq 0$

So A^{-1} exists.

$A^{-1} = \frac{1}{5}\begin{pmatrix} 7 & -9 \\ -1 & 2 \end{pmatrix}$

Step (2): Find $(A^{-1})^T$

$(A^{-1})^T = \frac{1}{5}\begin{pmatrix} 7 & -1 \\ -9 & 2 \end{pmatrix}$

Step (3): Find A^T

$A^T = \begin{pmatrix} 2 & 1 \\ 9 & 7 \end{pmatrix}$

Step (4): Find $(A^T)^{-1}$

Frat find its determinant:

$|A^T| = (2)(7) - (1)(9) = 14 - 9 = 5$

Now,

$(A^T)^{-1} = \frac{1}{5}\begin{pmatrix} 7 & -1 \\ -9 & 2 \end{pmatrix}$

Step (5): Compare results

$(A^T)^{-1} = \frac{1}{5}\begin{pmatrix} 7 & -1 \\ -9 & 2 \end{pmatrix} = (A^{-1})^T$

Hence proved:

$(A^T)^{-1} = (A^{-1})^T$

32. $4x^2 + 4px + (p+2) = 0$

To discuss the mature of roots, we examine the discriminant.

Step (1): Identify coefficients

$a = 4, b = 4p, c = p + 2$

Step (2): Find the discriminant D

$D = b^2 - 4ac$

$D = (4p)^2 - 4(4)(p+2)$

$D = 16p^2 - 16p - 32$

$D = 16(p^2 - p - 2)$

$D = 16(p-2)(p+1)$

Step (3): Discuss the nature of roots

(i) Real and distinct roots

When $D > 0$:

$(p-2)(p+1) > 0$

This happens when:

$p > 2$ or $p < -1$

(ii) Real and equal roots

When $D = 0$:

$(p-2)(p+1) = 0$

$p = 2$ or $p = -1$

(iii) Complex (non-real) roots

When $D < 0$:

$(p-2)(p+1) < 0$

$-1 < p < 2$

33. Given: -

- Length of the bridge (span) $= 40$ m $\Rightarrow$ Half span $= 20$ m

- Maximum height of the arch $= 15$ m
- Vertex is taken as $(0,0)$

Since the vertex is the highest point, the parabola opens downward.
Step (1): Write the general equation
With vertex at $(0,0)$, the equation of a downward-opening parabola is:
$y = ax^2\ (a < 0)$
Step (2): Use boundary conditions
At the ends of the bridge:
$x = \pm 20, y = -15$
Substitute in $y = ax^2$
$-15 = a(20)^2$
$-15 = 400a$
$a = -\frac{15}{400} = -\frac{3}{80}$
Final Equation of the Parabolic Arch:
$y = -\frac{3}{80}x^2$

34. Equation of the straight line
Given points:
$A(-5,7,-4), B(13,-5,2)$
Step (1): Direction vector of the line
$\overrightarrow{AB} = (13+5, -5-7, 2+4) = (18,-12,6)$
We can simplify by dividing by 6:
$\vec{d} = (3,-2,1)$
Step (2): Vector equation of the line
Position vector of point A :
$\vec{a} = -5\hat{\imath} + 7\hat{\jmath} - 4\hat{k}$
So, the vector equation is:
$\vec{r} = (-5\hat{\imath} + 7\hat{\jmath} - 4\hat{k}) + t(3\hat{\imath} - 2\hat{\jmath} + \hat{k})$
Step (3): Cartesian equation of the line
$\frac{x+5}{3} = \frac{y-7}{-2} = \frac{z+4}{1}$
Step (4): Point where the line crosses the xy-plane
On the xy-plane, $z = 0$.
From vector form:
$-4 + t(1) = 0 \Rightarrow t = 4$
Substitute $t = 4$:
$x = -5 + 3(4) = 7$
$y = 7 - 2(4) = -1$
Point of intersection with the xy-plane:
$(7,-1,0)$

35. Critical numbers of
$f(x) = x^{4/5}(x-4)^2$
Step (1): Differentiate
Using product rule:
$f'(x) = \frac{4}{5}x^{-1/5}(x-4)^2 + x^{4/5}\cdot 2(x-4)$
Factor common terms:
$f'(x) = x^{-1/5}(x-4)\left[\frac{4}{5}(x-4) + 2x\right]$
Simplify bracket:
$\frac{4}{5}x - \frac{16}{5} + 2x = \frac{14}{5}x - \frac{16}{5}$
So

$f'(x) = \frac{1}{5}x^{-1/5}(x-4)(14x-16)$
Step (2): Find critical numbers
Critical numbers occur when:
(1).$f'(x) = 0$
(2). $f'(x)$ is undefined but $f(x)$ is defined
(i) From numerator $= 0$:
$x - 4 = 0 \Rightarrow x = 4$
$14x - 16 = 0 \Rightarrow x = \frac{8}{7}$
(ii) From denominator undefined:
$x^{-1/5}$ undeflned at $x = 0$
But $f(0) = 0$, so $x = 0$ is also a critical number.

36. Given: -
$U = \log(x^3 + y^3 + z^3)$
Step (1): Find partial derivatives
Partial derivative w.r.t. x :
$\frac{\partial U}{\partial x} = \frac{1}{x^3+y^3+z^3} \cdot 3x^2 = \frac{3x^2}{x^3+y^3+z^3}$
Partial derivative w.r.t. y :
$\frac{\partial U}{\partial y} =$
$\frac{3y^2}{x^3+y^3+z^3}$
Partial derivative w.r.t. z :
$\frac{\partial U}{\partial z} = \frac{3z^2}{x^3+y^3+z^3}$
Step (2): Add the partial derivatives
$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = \frac{3(x^2+y^2+z^2)}{x^3+y^3+z^3}$

37. Given the probability mass function: -

X	1	2	3	4	5	6
$P(X=x)$	k	$2k$	$6k$	$5k$	$6k$	$10k$

Step (1): Find the value of k
Since the total probability is 1,
$k + 2k + 6k + 5k + 6k + 10k = 1$
$30k = 1$
$k = \frac{1}{30}$
Step (2): Find $P(2 < X < 6)$
The values of X satisfying $2 < X < 6$ are:
$X = 3,4,5$
$P(2 < X < 6) = P(3) + P(4) + P(5)$
$= 6k + 5k + 6k = 17k$
Substitute $k = \frac{1}{30}$:
$P(2 < X < 6) = \frac{17}{30}$

38. Given the probability density function (pdf): -
$f(x) = \begin{cases} kx(1-x)^{10}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$
Since X is a continuous random variable, the total probability must be 1.
Step (1): Use the normalization condition

$$\int_{-\infty}^{\infty} f(x)dx = 1$$
$$\Rightarrow \int_0^1 kx(1-x)^{10}dx = 1$$
$$k\int_0^1 x(1-x)^{10}dx = 1$$
Step (2): Evaluate the integral
Let $u = 1 - x$, then $x = 1 - u$ and $dx = -du$:
$\int_0^1 x(1-x)^{10}dx = \int_1^0 (1-u)u^{10}(-du) = \int_0^1 (1-u)u^{10}du$
$$= \int_0^1 (u^{10} - u^{11})du$$
$$= \left[\frac{u^{11}}{11} - \frac{u^{12}}{12}\right]_0^1$$
$$= \frac{1}{11} - \frac{1}{12} = \frac{1}{132}$$
Step (3): Find the value of k
$k \cdot \frac{1}{132} = 1$k
k=132

39. We prove the logical equivalence
$p \to q \equiv \neg p \vee q.$
Method (1): Truth Table Proof
Construct the truth table for both statements and compare their truth values.

p	q	$p \to q$	$\vec{p}$	$p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since the columns for $p \to q$ and $\neg p \vee q$ are identical in all cases, they are logically equivalent.
$p \to q \equiv \neg p \vee q$
Method (2): Logical Laws (Algebraic Proof)
Start with the implication and use standard logical identities:
$p \to q \equiv \neg(p \wedge \neg q)$ (definition of implication)
$\equiv \neg p \vee \neg\neg q$ (De Morgan's law)
$\equiv \neg p \vee q$ (double negation)
Thus,
$p \to q \equiv \neg p \vee q.$

40. Let the two given lines be
$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \text{ and } \frac{x-x_2}{l_2} = \frac{y-z_2}{m_2} = \frac{z-z_2}{n_2},$$
and suppose they lie in the same plane (ie, they are either intersecting or parallel).
The Cartesian equation of the plane containing both lines can be found in two standard ways.
Number of ways: $\boxed{2}$
Method (1): Determinant (Three vectors in the plane)
Idea
A plane can be determined by:
- one point in the plane, and
- two non-parallel drection vectors bing in the plane.
Steps:
- Take a point on the first line

$A(x_1, y_1, z_1)$
- Direction vectors of the two lines:
$d_1 = \langle l_1, m_1, n_1 \rangle, d_2 = \langle l_2, m_2, n_2 \rangle$
- For any point $P(x, y, z)$ on the plane, the vectors
$\overrightarrow{AP}, d_1$, and d_2 lie in the same plane
Hence their scalar triple product is zero: -

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Result: -
Expanding the determinant gives the Cartesian equation of the plane.
Method (2): Using the normal vector of the plane
Idea
The plane containing bath lines must have a normal vector perpendicular to both direction vector
Steps
- Direction vectors of the lines:
$\vec{d}_1 = \langle l_1, m_1, n_1 \rangle$; $\vec{d}_2 = \langle l_2, m_2, n_2 \rangle$
- Narmal vector of the plane

$$\vec{n} = \vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

- Take a point in the plane, say $A(x_1, y_1, z_1)$.
Plane equation
$n \cdot \langle x - x_1, y - y_1; z - z_1 \rangle = 0$
Expanding gives the Cartesian equation of the plane.

PART – IV

41. **(a)** We are given the system of linear equations:

$x - y + z = -9$
$2x - y + z = 4$
$3x - y + z = 6$
$4x - y + 2z = 7$

We will test consistency using the rank method.
Step (1): Write the augmented matrix

$$[A \mid B] = \begin{bmatrix} 1 & -1 & 1 & -9 \\ 2 & -1 & 1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \end{bmatrix}$$

Step (2): Perform row operations to find ranks
$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - 4R_1$

$$\begin{bmatrix} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 2 & -2 & 33 \\ 0 & 3 & -2 & 43 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$
$R_4 \to R_4 - 3R_2$

$$\begin{bmatrix} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 0 & -11 \\ 0 & 0 & 1 & -23 \end{bmatrix}$$

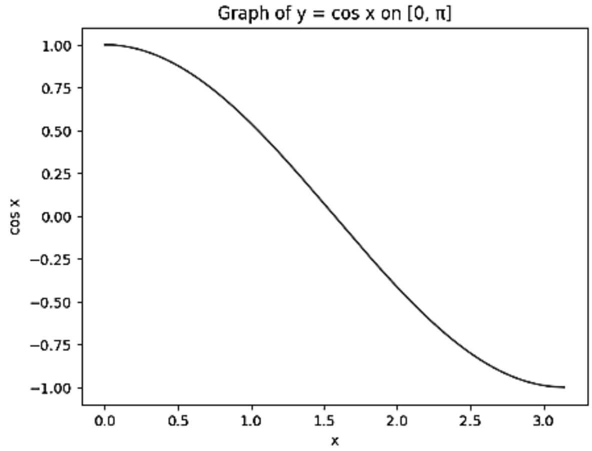
OR

(b) Given

$$2\cos\alpha = x + \frac{1}{x}, 2\cos\beta = y + \frac{1}{y}$$

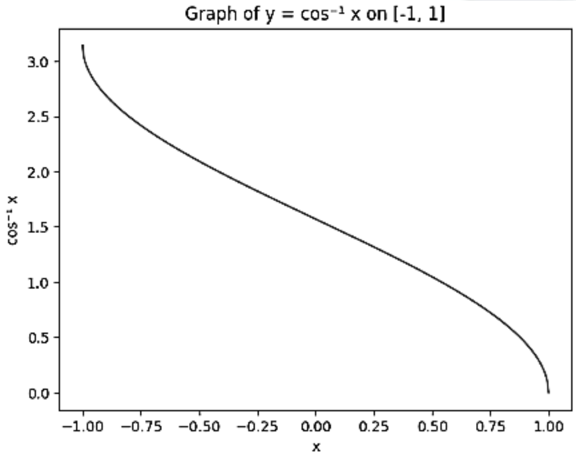
Step (1): Express x and y in exponential form
From Euler's formula:
$e^{i\theta} + e^{-i\theta} = 2\cos\theta$
Comparing, we get:
$x = e^{i\alpha}, \frac{1}{x} = e^{-i\alpha}$
$y = e^{i\beta}, \frac{1}{y} = e^{-i\beta}$
(i) Prove
$\frac{x^m}{y^n} - \frac{y^n}{x^m} = 2i\sin(m\alpha - n\beta)$
Proof:
$\frac{x^m}{y^n} = \frac{e^{im\alpha}}{e^{in\beta}} = e^{i(m\alpha - n\beta)}$
$\frac{y^n}{x^m} = \frac{e^{in\beta}}{e^{im\alpha}} = e^{-i(m\alpha - n\beta)}$
Subtract:
$\frac{x^m}{y^n} - \frac{y^n}{x^m} = e^{i(m\alpha - n\beta)} - e^{-i(m\alpha - n\beta)}$
Using Euler's identity:
$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$
$\frac{x^m}{y^n} - \frac{y^n}{x^m} = 2i\sin(m\alpha - n\beta)$
(ii) Prove
$x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$
Proof:
$x^m y^n = e^{im\alpha} e^{in\beta} = e^{i(m\alpha + n\beta)}$
$\frac{1}{x^m y^n} = e^{-i(m\alpha + n\beta)}$
Add:
$x^m y^n + \frac{1}{x^m y^n} = e^{i(m\alpha + n\beta)} + e^{-i(m\alpha + n\beta)}$
Using Euler's identity:
$e^{i\theta} + e^{-i\theta} = 2\cos\theta$
$x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$

42. (a) (i) Graph of $y = \cos x$ on $[0, \pi]$
- Starts at (0,1)
- Decreases smoothly
- Ends at $(\pi, -1)$
- It is monotonically decreasing on $[0, \pi]$



(ii) Graph of $y=\cos^{-1} x$ on $[-1,1]$

- Starts at $(-1, \pi)$
- Decreases smoothly
- Ends at $(1,0)$
- Range is $[0, \pi]$
- This graph is the reflection of $y=\cos x$ (restricted to $[0, \pi]$) in the line $y=x$



OR

(b) To find the equation of the circle passing through the points $(1,1),(2,-1),(3,2)$:

Step (1): Write the general equation of a circle

$x^2+y^2+2gx+2fy+c=0$

Step (2): Substitute the given points

Using point (1,1):

$1+1+2g+2f+c=0$

$2g+2f+c=-2$

Using point $(2,-1)$:

$4+1+4g-2f+c=0$

$4g-2f+c=-5$

Using point (3,2):

$9+4+6g+4f+c=0$

$6g+4f+c=-13$

Step (3): Solve the equations

Subtract (1) from (2):

$(4g-2f+c)-(2g+2f+c)=-5+2$

$2g-4f=-3\ (A)$

Subtract (2) from (3):

$(6g+4f+c)-(4g-2f+c)=-13+5$

$2g+6f=-8\ (B)$

Solve (A) and (B):

From (A):

$g-2f=-\frac{3}{2}$

From (B):

$g+3f=-4$

Subtract:

$5f=-\frac{5}{2}\Rightarrow f=-\frac{1}{2}$

Substitute in $g+3f=-4$:

$g+3\left(-\frac{1}{2}\right)=-4\Rightarrow g=-\frac{5}{2}$

Substitute g,f into (1):

$2\left(-\frac{5}{2}\right)+2\left(-\frac{1}{2}\right)+c=-2\Rightarrow c=4$

Step (4): Write the equation of the circle

$x^2+y^2-5x-y+4=0$

43. (a) Step (1): Set up coordinates

- Take the end of the pipe as the origin $O(0,0)$.
- Let the x-axis be horizontal (direction of water flow).
- Let the y-axis be vertical downwards (mince the water falls down).

Then:

- Vertex of the parabola is at $(0,0)$.
- Water follows a parabolic path opening downward, so its equation is

$y=ax^2$

Step (2): Use the given point

At a position 2.5 m below the pipe, the water has moved 3 m horizontally.

So the point is

$(x,y)=(3,2.5)$

Substitute in $y=ax^2$.

$2.5=a(3)^2$

$2.5=9a$

$a=\frac{5}{18}$

Hence, the equation of the path is:

$y=\frac{5}{18}x^2$

Step (3): Find where the water strikes the ground

The pipe is 7.5m above the ground, so the water strikes the ground when:

$y=7.5$

Substitute in the equation:

$7.5=\frac{5}{18}x^2$

Multiply bath sides by 18:

$135=5x^2$

$x^2=27$

$x=\sqrt{27}=3\sqrt{3}$

OR.

(b) Step (1): Define two-unit vectors

Let

- $\vec{a}$ be a unit vector making an angle α with the positive x-axis.
- $\vec{b}$ be a unit vector making an angle β with the positive x-axis.

Then their components are: -

$\vec{a}=\cos\alpha\hat{\imath}+\sin\alpha\hat{\jmath}$

$\vec{b}=\cos\beta\hat{\imath}+\sin\beta\hat{\jmath}$

Step (2): Find the dot product using components

$\vec{a}\cdot\vec{b}=(\cos\alpha\hat{\imath}+\sin\alpha\hat{\jmath})\cdot(\cos\beta\hat{\imath}+\sin\beta\hat{\jmath})$

$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$ (1)
Step (3): Find the dot product using angle between vectors
The angle between $\vec{a}$ and $\vec{b}$ is $|\alpha - \beta|$.
Since both are unit vectors:
$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\alpha - \beta) = \cos(\alpha - \beta)$
Step (4): Compare (1) and (2)
$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
Step (5): Replace β by $-\beta$
Using:
$\cos(-\beta) = \cos\beta, \sin(-\beta) = -\sin\beta$
$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

44. (a) Given point: -
$A(0,1,-5)$
Given lines (parallel to the required plane):
(1). $\vec{r} = (\hat{\imath} + 2\hat{\jmath} - 4\hat{k}) + s(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k})$
$\Rightarrow$ direction vector
$d_1 = (2,3,6)$
(2).$\vec{r} = (\hat{\imath} - 3\hat{\jmath} + 5\hat{\imath}) + t(\hat{\imath} + \hat{\jmath} - \hat{k})$
$\Rightarrow$ direction vector
$d_2 = (1,1,-1)$
Step (1): Find the normal vector of the plane
The plane is parallel to both lines, so its normal vector is perpendicular to bath direction vectors:
$\hat{n} = \hat{d}_1 \times \hat{d}_2$
$$\hat{n} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix} = -9\hat{\imath} + 8\hat{\jmath} - \hat{k}$$
So, a normal vector is
$n = (-9,8,-1)$
Step (2): Vector equation of the plane
A plane through point $A(0,1,-5)$ and parallel to vectors $\dot{d}_1$ and $\dot{d}_2$ is
$\vec{r} = (0\hat{\imath} + 1\hat{\jmath} - 5\hat{k}) + \lambda(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}) + \mu(\hat{\imath} + \hat{\jmath} - \hat{k})$
Step)3): Cartesian equation of the plane
Using
$\bar{n} \cdot (\bar{r} - \bar{a}) = 0$
where $\vec{a} = (0,1,-5)$.
$-9(x - 0) + 8(y - 1) - (z + 5) = 0$
$9x - 8y + z + 13 = 0$

OR

(b) Evaluate
$$I = \int_{-x}^{x} \frac{\cos^2 x}{1 + \alpha^x} dx$$
Step (1): Use the substitution $x \to -x$
Let
$$I = \int_{-\infty}^{x} \frac{\cos^2 x}{1 + \alpha^x} dx$$
Replace x by -x:
$$I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1 + a^{-x}} dx$$
since $\cos^3(-x) = \cos^2 x$.
Now,
$$\frac{1}{1 + a^{-x}} = \frac{a^x}{1 + a^x}$$
So,

$I=\int_{-\pi}^{\pi}\frac{a^{x}\cos^{2}x}{1+a^{x}}dx$

Step (2): Add the two expressions for I

$2I=\int_{-\pi}^{\pi}\cos^{2}x\left(\frac{1}{1+\alpha^{x}}+\frac{\alpha^{x}}{1+\alpha^{x}}\right)dx$

$2I=\int_{-\pi}^{\pi}\cos^{2}xdx$

Step (3): Evaluate the remaining integral

$\int_{-\pi}^{\pi}\cos^{2}xdx$

Use $\cos^{2}x=\frac{1\tan 2x}{2}$;

$\int_{-\pi}^{\pi}\cos^{2}xdx=\frac{1}{2}\int_{-\pi}^{\pi}1dx+\frac{1}{2}\int_{-\pi}^{\pi}\cos 2xdx$

$=\frac{1}{2}(2\pi)+0=\pi$

Step (4): Final result

$2I=\pi\ \Rightarrow\ I=\frac{\pi}{2}$

45. (a) Step (1): Set up coordinates

Let the intersection be the arraign.

- Car is moving east along the 2 -axis

$\Rightarrow x=$ distance of car from intersection

Given at the instant: $\mathrm{r}=0.8$ km

- Jeep is moving south alang the y-axis toward the intersection

$\Rightarrow y=$ distance of jeep from intersection

Given at the instant: $y=0.6$ km

- Distance between jeep and car $=r$

By Pythagoras:

$r^{2}=x^{2}+y^{2}$

Step (2): Differentiate w.r.t. time t

$2r\frac{dr}{dt}=2x\frac{dz}{dt}+2y\frac{dy}{dt}$

Cancel 2:

$r\frac{dr}{dt}=x\frac{dx}{dt}+y\frac{dy}{dt}$

Step (3): Substitute known values

Frast find r :

$r=\sqrt{0.8^{2}+0.6^{2}}=\sqrt{0.64+0.36}=1$ km

Given:

- $\frac{dr}{dt}=20$ km/hr (distance increasing)

- Jeep speed $=60$ km/hr, moving south $-\frac{dy}{dt}=-60$ km/hr

- Car speed $=\frac{dr}{dt}=v$ (to be found)

Substitute:

$1(20)=0.8(v)+0.6(-60)$

Step (4): Solve

$20=0.8v-36$

$0.8v=56$

$v=70$ km/hr

OR

(b) We are given the region bounded by

- the x-axis

- the curve $y=|\cos x|$

- the lines $x = 0$ and $x = \pi$

Step 1: Understand $y = |\cos x|$ on $[0, \pi]$

- $\cos x \geq 0$ on $\left[0, \frac{\pi}{2}\right]$
- $\cos x \leq 0$ on $\left[\frac{\pi}{2}, \pi\right]$

So,

$$|\cos x| = \begin{cases} \cos x, & 0 \leq x \leq \dfrac{\pi}{2} \\ -\cos x, & \dfrac{\pi}{2} \leq x \leq \pi \end{cases}$$

Step (2): Write the area integral

$$\text{Area } = \int_0^{\pi} |\cos x| dx = \int_0^{\pi/2} \cos x dx + \int_{\pi/2}^{\pi} (-\cos x) dx$$

Step (3): Evaluate the integrals

$$\int_0^{\pi/2} \cos x dx = \sin x|_0^{\pi/2} = 1$$

$$\int_{\pi/2}^{\pi} (-\cos x) dx = -\sin x|_{\pi/2}^{\pi} = -(0-1) = 1$$

Step 4: Total area

Area $= 1 + 1 = 2$

46. (a) Step (1): Define variables

- Let the side of the ariginal square sheet be S.

$S^2 = 196 \Longrightarrow S = 14$ units

- Let the side of each small square cut from the carners be x.

After cutting and folding the box will have :-

- Length and width: $14 - 2x$
- Height x

Step (2): Express the volume V

$V = \text{ length } \times \text{ width } \times \text{ height } = (14 - 2x)^2 \cdot x$

$V(x) = x(14 - 2x)^2$

Step (3): Expand the expression

$V(x) = x(196 - 56x + 4x^2) = 196x - 56x^2 + 4x^3$

$V(x) = 4x^3 - 56x^2 + 196x$

Step (4): Differentiate $V(x)$ to find maximum

$$\frac{dV}{dx} = 12x^2 - 112x + 198$$

Set derivative to zero for critical points

$12x^2 - 112x + 196 = 0$

Divide whole equation by 4 to simplify:

$3x^2 - 28x + 49 = 0$

Step (5): Solve quadratic equation

$$x = \frac{28 \pm \sqrt{(-28)^2 - 4 \cdot 3 \cdot 49}}{2 \cdot 3} = \frac{28 \pm \sqrt{784 - 588}}{6} = \frac{28 \pm \sqrt{196}}{6} = \frac{28 \pm 14}{6}$$

Two solutions:

(1). $x = \frac{2x+14}{6} = \frac{p}{8} = 7$

(2). $x = \frac{2x-14}{6} = \frac{14}{6} = \frac{7}{3}$

Step (6): Choose the feasible solution

- If $x = 7$, then $14 - 2x = 0$, which gives zero length/width - not feasible.
- So, the maximum volume occurs at:

$$x = \frac{7}{3}$$

Step (7): Conclusion

The length of the side of the square to cut for maximum volume is

$\frac{7}{3}$ units

OR

(b) We are asked to solve the first-order differential equation:

$$M\frac{dV}{dt} = F - kV$$

with initial condition $V(0) = 0$, and show that

$$V = \frac{F}{k}\left(1 - e^{-kt/M}\right)$$

Step (1): Rewrite the equation in standard form

Divide both sides by M :

$$\frac{dV}{dt} = \frac{F - kV}{M} = \frac{F}{M} - \frac{k}{M}V$$

$$\frac{dV}{dt} + \frac{k}{M}V = \frac{F}{M}$$

This is a linear first-order ODE in V.

Step (2): Find the integrating factor

For a linear ODE of the form:

$$\frac{dV}{dt} + P(t)V = Q(t)$$

the integrating factor is:

$$\mu(t) = e^{\int P(t)dt}$$

Here, $P(t) = \frac{k}{M}$ (constant), so:

$$\mu(t) = e^{\int k/M dt} = e^{(k/M)t}$$

Step (3): Multiply the ODE by the integrating factor

$$e^{(k/M)t}\frac{dV}{dt} + \frac{k}{M}e^{(k/M)t}V = \frac{F}{M}e^{(k/M)t}$$

Left-hand side is derivative of $\left(Ve^{(k/M)t}\right)$:

$$\frac{d}{dt}\left(Ve^{(k/M)t}\right) = \frac{F}{M}e^{(k/M)t}$$

Step (4): Integrate both sides

$$\int \frac{d}{dt}\left(Ve^{(k/M)t}\right)dt = \int \frac{F}{M}e^{(k/M)t}dt$$

$$Ve^{(k/M)t} = \frac{F}{M}\cdot\frac{M}{k}e^{(k/M)t} + C$$

$$Ve^{(k/M)t} = \frac{F}{k}e^{(k/M)t} + C$$

Step 5: Solve for V

$$V = \frac{F}{k} + Ce^{-(k/M)t}$$

Step (6): Apply initial condition $V(0) = 0$

$$0 = \frac{F}{k} + C \Rightarrow C = -\frac{F}{k}$$

Step (7): Final solution

$$V = \frac{F}{k}\left(1 - e^{-(k/M)t}\right)$$

47. **(a)** Step (1): Newton's Law of Cooling

The law states:

$$\frac{dT}{dt} = -k(T - T_r)$$

Where:

- $T(t)$ = body temperature at time t
- $T_r = 50^\circ F$ = room temperature
- k = cooling constant
- t = time since death

The solution is:
$T(t)=T_r+(T_0-T_r)e^{-kt}$
Where $T_0=98.6^{\circ}\mathrm{F}=$ normal body temperature at death.
Step (2): Introduce known temperatures
- At $t_1=?$ (time of death) $\rightarrow T_0=98.6$
- At $t=8$ p.m., $T_1=70^{\circ}\mathrm{F}$
- At $t=10\mathrm{p.m.}, T_2=60^{\circ}\mathrm{F}$
Let t be the time in hours since death. Then:
$T(t)=50+(98.6-50)e^{-kt}=50+48.6e^{-kt}$
Step (3): Use the two measurements to find k
Let's denote:
- At 8 p.m, $t_1=?\rightarrow T_1=70$

$70=50+48.6e^{-kt_1}\Longrightarrow 20=48.6e^{-kt_1}\Longrightarrow e^{-kt_1}=\frac{20}{48.6}\approx 0.4115$

- Two hours later, $t_2=t_1+2\rightarrow T_2=60$
$60=50+48.6e^{-k(t_1+2)}\Longrightarrow 10=48.6e^{-kt_1}e^{-2k}$
Substitute $e^{-kt_2}=0.4115$:

$10=48.6\cdot 0.4115\cdot e^{-2k}\Longrightarrow 10=20\cdot e^{-2k}\Longrightarrow e^{-2k}=\frac{10}{20}=0.5$

$-2k=\ln 0.5\Longrightarrow k=\frac{\ln 2}{2}\approx 0.3466\mathrm{hr}^{-1}$

(Check: $e^{-2k}=0.5$ V)
Step (4): Find time since death until 8 p.m.

$e^{-kt_1}=0.4115\Longrightarrow t_1=-\frac{\ln 0.4115}{k}$

$\ln 0.4115\approx -0.888$ and $k=0.3466$

$t_1=\frac{0.888}{0.3466}\approx 2.56$ hours

Step (5): Convert to hours and minutes
$0.56\mathrm{hr}=0.56\times 60\text{ min}\approx 33.6\text{ min}\approx 34\text{ min}$
So, time since death until 8 p.m. » 2 hr 34 min
Time of death $=8$ p.m. $-$ 2hr34 min $\approx 5:26$ p.m.

OR

(b) Step (1): Define the random variable
Let $X=$ number of heads when 3 fair coins are tossed.
- Possible values of X: 0,1,2,3
Step (2): Find the probability mass function (PMF)
All outcomes are equally likely, total outcomes $=2^3=8$.
List of outcomes and number of heads:

Outcome	#Heads
TTT	0
HTT	1
THT	1
TTH	1
HHT	2
HTH	2
THH	2
HHH	3
Count probabilities:	

$P(X=0)$ $=\frac{1}{8}, P(X=1)$ $=\frac{3}{8}, P(X=2)$ $=\frac{3}{8}, P(X=3)$ $=\frac{1}{8}$

PMF:

$$p(x)=\begin{cases}\frac{1}{8}, & x=0\\ \frac{3}{8}, & x=1\\ \frac{3}{3}, & x=2\\ \frac{1}{8}, & x=3\\ 0, & \text{otherwise}\end{cases}$$

Step (3): Find the mean (expected value)

$$E[X]=\sum x\cdot P(X=x)$$

$$E[X]=0\cdot\frac{1}{8}+1\cdot\frac{3}{8}+2\cdot\frac{3}{8}+3\cdot\frac{1}{8}=0+\frac{3}{8}+\frac{6}{8}+\frac{3}{8}=\frac{12}{8}=1.5$$

Mean: $\mu=1.5$

Step (4): Find the variance

$\operatorname{Var}(X)=E[X^2]-(E[X])^2$

First, compute $E[X^2]$:

$$E[X^2]=0^2\cdot\frac{1}{8}+1^2\cdot\frac{3}{8}+2^2\cdot\frac{3}{8}+3^2\cdot\frac{1}{8}=0+\frac{3}{8}+\frac{12}{8}+\frac{9}{8}=\frac{24}{8}=3$$

$\operatorname{Var}(X)=3-(1.5)^2=3-2.25=0.75$

Variance: $\sigma^2=0.75$

Step (5): Verify using Binomial Distribution

- For binomial distribution: $X\sim\operatorname{Bin}(n=3,p=0.5)$

$$P(X=x)=\binom{3}{x}(0.5)^x(0.5)^{3-x}=\binom{3}{x}(0.5)^3$$

Compute probabilities:

- $x=0$: $P=\binom{3}{0}(0.5)^3=1/8$
- $x=1$: $P=\binom{3}{1}(0.5)^3=3/8$
- $x=2$: $P=\binom{3}{2}(0.5)^3=3/8$
- $x=3$: $P=\binom{3}{3}(0.5)^3=1/8$ Matches PMF above.

Mean and Variance of Binomial:

$E[X]=np=3\cdot 0.5=1.5, \operatorname{Var}(X)=np(1-p)=3\cdot 0.5\cdot 0.5=0.75$

Perfect match.

Answer: -

- PMF:

$$p(x)=\begin{cases}1/8, & x=0\\ 3/8, & x=1\\ 3/8, & x=2\\ 1/8, & x=3\end{cases}$$

- Mean: $E[X]=1.5$
- Variance: $\operatorname{Var}(X)=0.75$
- Verified by Binomial distribution $B(3,0.5)$.