PART - I

1. (b)
2. (d)
3. (b)
4. (a)
5. (a)
6. (c)
7. (b)
8. (d)
9. (a)
10. (d)
11. (a)
12. (a)
13. (c)
14. (c)
15. (b)
16. (d)
17. (b)
18. (d)
19. (a)
20. (b)

PART - II

21. Given $|z| = 2$

Using triangle inequality: -

$|z + 3 + 4i| \leq |z| + |3 + 4i| = 2 + 5 = 7$

Using reverse triangle inequality: -

$|z + 3 + 4i| \geq ||3 + 4i| - |z|| = |5 - 2| = 3$

$\boxed{3 \leq |z + 3 + 4i| \leq 7}$

22. From the quadratic equation:

$x^2 + nx + n = 0$

Sum and product of roots:

$p + q = -n, pq = n$

Now,

$$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} = \frac{p+q}{\sqrt{pq}}$$

Substitute values:

$$= \frac{-n}{\sqrt{n}} = -\sqrt{n}$$

Therefore,

$$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{n} = -\sqrt{n} + \sqrt{n} = 0$$

23. Line: -

$y = 4x + c \Rightarrow 4x - y + c = 0$

Circle: -

$x^2 + y^2 = 9$ (center $(0,0)$, radius $= 3$)

For tangency, distance from center to line = radius:

$$\frac{|c|}{\sqrt{4^2 + 1}} = 3 \Rightarrow \frac{|c|}{\sqrt{17}} = 3$$

$|c| = 3\sqrt{17}$

$\boxed{c = \pm 3\sqrt{17}}$

24. Volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

Decrease in radius:

$dr = -0.1 \text{ cm}, r = 10 \text{ cm}$

Using differentials:

$dV = 4\pi r^2 dr$

$dV = 4\pi(10)^2(-0.1) = -40\pi$

So, the approximate decrease in volume is:

$\boxed{40\pi \text{ cm}^3}$

25. Evaluate:

$$\int_b^\infty \frac{1}{a^2 + x^2} dx \ (a > 0, b \in \mathbb{R})$$

We know:

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

So,

$$\int_b^\infty \frac{dx}{a^2 + x^2} = \left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]_b^\infty$$

Since $\tan^{-1}(\infty) = \frac{\pi}{2}$,

$$= \frac{1}{a}\left(\frac{\pi}{2} - \tan^{-1}\frac{b}{a}\right)$$

$$\int_b^\infty \frac{dx}{a^2 + x^2} = \frac{1}{a}\left(\frac{\pi}{2} - \tan^{-1}\frac{b}{a}\right)$$

26. Normal vector:

$\vec{n} = 3\hat{\imath} - 4\hat{\jmath} + 5\hat{k}, |\vec{n}| = \sqrt{50} = 5\sqrt{2}$

Unit normal:

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{1}{5\sqrt{2}}(3\hat{\imath} - 4\hat{\jmath} + 5\hat{k})$$

Plane at distance 7 from origin:

$\vec{r} \cdot \hat{n} = 7$

$\Rightarrow \vec{r} \cdot (3\hat{\imath} - 4\hat{\jmath} + 5\hat{k}) = 35\sqrt{2}$

27. Given Boolean matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

For Boolean matrices:

- $A \vee B$: entrywise OR
- $A \wedge B$: entrywise AND

$A \vee B$

$$\begin{bmatrix} 0 \vee 1 & 1 \vee 1 \\ 1 \vee 0 & 1 \vee 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$A \wedge B$

$$\begin{bmatrix} 0 \wedge 1 & 1 \wedge 1 \\ 1 \wedge 0 & 1 \wedge 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A \vee B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, A \wedge B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

28. Let

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Transpose of A :

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Now,

$A^T A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

29. Given curve:

$y = x^2 + 3x - 2$

Differentiate:

$\frac{dy}{dx} = 2x + 3$

At $x = 1$:

slope $m = 2(1) + 3 = 5$

Equation of tangent at (1,2):

$y - 2 = 5(x - 1)$

$\boxed{y = 5x - 3}$

30. $e^{\cos\theta + i\sin\theta} = e^{\cos\theta} e^{i\sin\theta}$

Using Euler's formula:

$e^{i\phi} = \cos\phi + i\sin\phi$

$= e^{\cos\theta}(\cos(\sin\theta) + i\sin(\sin\theta))$

Hence, in $a + ib$ form:

$\boxed{a = e^{\cos\theta}\cos(\sin\theta), b = e^{\cos\theta}\sin(\sin\theta)}$

PART - III

31. Axis parallel to **y**-axis ⇒ parabola is of the form

$(x - h)^2 = 4a(y - k)$

Vertex $(h, k) = (-1, -2)$, so:

$(x + 1)^2 = 4a(y + 2)$

Since it passes through $(3,6)$:

$(3 + 1)^2 = 4a(6 + 2)$

$16 = 32a \Rightarrow a = \frac{1}{2}$

Substitute a :

$(x + 1)^2 = 2(y + 2)$

$(x + 1)^2 = 2(y + 2)$

32. For an elliptical orbit with the Sun at one focus:

Maximum distance $= a + c$

Minimum distance $= a - c$

Given: -

$a + c = 152 \times 10^6$ km, $a - c = 94.5 \times 10^6$ km

Subtract: -

$2c = (152 - 94.5) \times 10^6 = 57.5 \times 10^6$ km

So, distance between the two foci: -

$2c = 57.5 \times 10^6 = 575 \times 10^5$ km

33. Given:

$\frac{\pi}{2} < \cos^{-1}(3x - 1) < \pi$

Since $\cos^{-1} x$ is decreasing,

$\cos\pi < 3x - 1 < \cos\frac{\pi}{2}$

$-1 < 3x - 1 < 0$

Add 1 throughout:

$0 < 3x < 1$

Divide by 3:

$0 < x < \frac{1}{3}$

$0 < x < \frac{1}{3}$

34. Given line:
$$\frac{x+3}{2}=\frac{y-1}{2}=-z$$
Let each be equal to t :
$x=2t-3, y=2t+1, z=-t$
So, direction ratios are:
$(2,2,-1)$
Magnitude:
$$\sqrt{2^2+2^2+(-1)^2}=\sqrt{9}=3$$
Hence, direction cosines:
$$\cos\alpha=\frac{2}{3}, \cos\beta=\frac{2}{3}, \cos\gamma=-\frac{1}{3}$$
Therefore, the angles with the coordinate axes are:
$$\alpha=\cos^{-1}\left(\frac{2}{3}\right), \beta=\cos^{-1}\left(\frac{2}{3}\right), \gamma=\cos^{-1}\left(-\frac{1}{3}\right)$$

35. Let
$f(x)=x^{2/3}$
Choose a nearby perfect cube: $x_0=125$
$f(125)=125^{2/3}=(5^3)^{2/3}=25$
Derivative:
$$f'(x)=\frac{2}{3}x^{-1/3}$$
$$f'(125)=\frac{2}{3}\cdot\frac{1}{5}=\frac{2}{15}$$
Linear approximation:
$f(x)\approx f(x_0)+f'(x_0)(x-x_0)$
For $x=123$:
$$f(123)\approx 25+\frac{2}{15}(123-125)=25-\frac{4}{15}=\frac{371}{15}\approx 24.73$$
$(123)^{2/3}\approx 24.73$

36. $x\cos y dy=e^x x(\log x+1)dx$
Divide both sides by x :
$\cos y dy=e^x(\log x+1)dx$
Integrate both sides:
$$\int \cos y dy=\int e^x(\log x+1)dx$$
LHS:
$\sin y$
RHS:
$$\int e^x(\log x+1)dx=e^x\log x+C$$

37. Given
$$F(\alpha)=\begin{bmatrix}\cos\alpha & 0 & \sin\alpha\\ 0 & 1 & 0\\ -\sin\alpha & 0 & \cos\alpha\end{bmatrix}$$
Compute $F(-\alpha)$:
$$F(-\alpha)=\begin{bmatrix}\cos\alpha & 0 & -\sin\alpha\\ 0 & 1 & 0\\ \sin\alpha & 0 & \cos\alpha\end{bmatrix}$$
Now multiply:
$$F(\alpha)F(-\alpha)=\begin{bmatrix}\cos^2\alpha+\sin^2\alpha & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \cos^2\alpha+\sin^2\alpha\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I$$
Since $F(\alpha)F(-\alpha)=I$,.
$\boxed{[F(\alpha)]^{-1}=F(-\alpha)}$

38. Truth table:

P	q	p → q	q → p
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Since the truth values of $p \rightarrow q$ and $q \rightarrow p$ differ (e.g., when $p = T, q = F$), they are not equivalent.

39. First find z :

$z = (2 + 3i)(1 - i) = 2 - 2i + 3i - 3i^2 = 2 + i + 3 = 5 + i$

Now,

$z^{-1} = \dfrac{1}{5 + i}$

Multiply numerator and denominator by the conjugate $(5 - i)$:

$z^{-1} = \dfrac{5 - i}{5^2 + 1^2} = \dfrac{5 - i}{26}$

40. Given:

$a + b + c = 0$ and $a, b, c \in \mathbb{Q}$

Consider the equation:

$(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$

Using $a + b + c = 0$:

$b + c = -a, c + a = -b, a + b = -c$

Substitute:

$(-a - a)x^2 + (-b - b)x + (-c - c) = 0$

$-2ax^2 - 2bx - 2c = 0$

Divide by -2:

$ax^2 + bx + c = 0$

Since a, b, c are rational numbers, the coefficients of the quadratic are rational. Hence, its roots are rational numbers.

$\boxed{\text{Roots are rational}}$

PART - IV

41. **(a)** Solve:

$z^3 + 8i = 0 \Rightarrow z^3 = -8i$

Write $-8i$ in polar form:

$-8i = 8\left(\cos\dfrac{3\pi}{2} + i\sin\dfrac{3\pi}{2}\right)$

Cube roots:

$z = 2\left(\cos\dfrac{3\pi/2 + 2k\pi}{3} + i\sin\dfrac{3\pi/2 + 2k\pi}{3}\right), k = 0,1,2$

Thus,

- $k = 0 : z = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2i$
- $k = 1 : z = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = -\sqrt{3} - i$
- $k = 2 : z = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = \sqrt{3} - i$

OR

(b) Given:

$(1 + x + xy^2)\dfrac{dy}{dx} + (y + y^3) = 0$

Rewrite:

$(y + y^3)dx + (1 + x + xy^2)dy = 0$

Divide by $(1 + y^2)$:

$ydx + (1 + x)dy = 0$

Now it is exact.

Integrate:

$$\int ydx = xy, \int (1+x)dy = y + xy$$

Hence,

$xy + y = C$

$\boxed{y(x+1) = C}$

42. (a) Let

$\vec{a} = (\cos\alpha, \sin\alpha), \vec{b} = (\cos\beta, \sin\beta)$

Then,

$\vec{a}\cdot\vec{b} = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Also, angle between $\vec{a}$ and $\vec{b}$ is $\alpha - \beta$, so

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos(\alpha-\beta)$

Since $|\vec{a}| = |\vec{b}| = 1$,

$\vec{a}\cdot\vec{b} = \cos(\alpha-\beta)$

Thus,

$\boxed{\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta}$

OR

(b) Given pdf:

$$f(x) = \begin{cases} k, & 200 \le x \le 600 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find k

$$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{200}^{600} kdx = 1$$

$$k(600-200) = 1 \Rightarrow k = \frac{1}{400}$$

(ii) Distribution function $F(x)$

$$F(x) = P(X \le x) = \begin{cases} 0, & x < 200 \\ \int_{200}^{x} \frac{1}{400}dt = \frac{x-200}{400}, & 200 \le x \le 600 \\ 1, & x > 600 \end{cases}$$

(iii) $P(300 \le X \le 500)$

$$P(300 \le X \le 500) = \int_{300}^{500} \frac{1}{400}dx = \frac{500-300}{400} = \frac{1}{2}$$

$$\boxed{k = \frac{1}{400}, F(x) = \frac{x-200}{400}(200 \le x \le 600), P = \frac{1}{2}}$$

43. (a) Step (1): Group terms and complete squares

$18(x^2 - 8x) + 12(y^2 + 4y) + 120 = 0$

$18[(x-4)^2 - 16] + 12[(y+2)^2 - 4] + 120 = 0$

$18(x-4)^2 + 12(y+2)^2 - 288 - 48 + 120 = 0$

$18(x-4)^2 + 12(y+2)^2 = 216$

Divide by 216

$$\frac{(x-4)^2}{12} + \frac{(y+2)^2}{18} = 1$$

Step (2): Identify conic

Both terms positive → Elipse

Major axis along y-axis (larger denominator = 18)

Step (3): Centre

$\boxed{(4, -2)}$

Step (4): Vertices

$a^2 = 18 \Rightarrow a = 3\sqrt{2}, b^2 = 12 \Rightarrow b = 2\sqrt{3}$

Vertices along y-avic

$(4, -2 \pm 3\sqrt{2})$

Step (5): Foci

$c^2 = a^2 - b^2 = 18 - 12 = 6 \Rightarrow c = \sqrt{6}$
Foci
$(4, -2 \pm \sqrt{6})$

OR

(b) Let $\cos^{-1} x = \alpha, \cos^{-1} y = \beta$ and $\cos^{-1} z = \gamma$. Then $\cos\alpha = x, \cos\beta = y, \cos\gamma = z$
$\Rightarrow \cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$
$\Rightarrow \cos^{-1} x + \cos^{-1} y = \pi - \cos^{-1} z$
$\Rightarrow \alpha + \beta = \pi - \gamma$
$\Rightarrow \cos(\alpha + \beta) = \cos(\pi - \gamma)$
$\Rightarrow \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta = -\cos\gamma$
$\Rightarrow \cos\alpha \cdot \cos\beta - \sqrt{(1 - \cos^2\alpha)}\sqrt{(1 - \cos^2\beta)} = -\cos\gamma$
$\Rightarrow xy - \sqrt{(1 - x^2)}\sqrt{(1 - x^2)} = -z$
$\Rightarrow xy + z = \sqrt{(1 - x^2)}\sqrt{1 - x^2}$
$\Rightarrow (xy + z)^2 = (1 - x^2)(1 - y^2)$
$\Rightarrow x^2y^2 + z^2 + 2xyz = 1 - y^2 - x^2 + x^2y^2$
$\Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$

44. (a) Given points: $(-6,8), (-2,-12), (3,8)$
Equation: $y = ax^2 + bx + c$
Form equations:
$36a - 6b + c = 8, 4a - 2b + c = -12, 9a + 3b + c = 8$
Gaussian elimination:
From 2nd: $c = -12 - 4a + 2b$
Substitute in 1st: $36a - 6b + (-12 - 4a + 2b) = 8 \Longrightarrow 8a - b = 5$
Substitute in 3rd: $9a + 3b + (-12 - 4a + 2b) = 8 \Longrightarrow a + b = 4$
Solve: $a = 1, b = 3, c = -10$
Equation: $y = x^2 + 3x - 10$
Check $P(7,60)$: $7^2 + 3 \cdot 7 - 10 = 60$
$\boxed{\text{Yes, he will meet his friend.}}$

OR

(b) Step (1): Slopes of tangents
For a curve $F(x, y) = 0$, slope of tangent:
$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$
Ellipse: $F(x, y) = x^2 + 4y^2 - 8 = 0$
$$F_x = 2x, F_y = 8y \Longrightarrow m_1 = -\frac{2x}{8y} = -\frac{x}{4y}$$
Hyperbola: $G(x, y) = x^2 - 2y^2 - 4 = 0$
$$G_x = 2x, G_y = -4y \Longrightarrow m_2 = -\frac{2x}{-4y} = \frac{x}{2y}$$
Step (2): Check orthogonality
Two curves intersect orthogonally if $m_1m_2 = -1$:
$$m_1m_2 = \left(-\frac{x}{4y}\right)\left(\frac{x}{2y}\right) = -\frac{x^2}{8y^2}$$
At intersection points, solve simultaneously:
$x^2 + 4y^2 = 8, x^2 - 2y^2 = 4$
Subtract:
$(x^2 + 4y^2) - (x^2 - 2y^2) = 6y^2 = 4 \Longrightarrow y^2 = 2/3$
Then $x^2 + 4(2/3) = 8 \Longrightarrow x^2 + 8/3 = 8 \Longrightarrow x^2 = 16/3$
So at intersection points:
$$m_1m_2 = -\frac{16/3}{8 \cdot (2/3)} = -\frac{16/3}{16/3} = -1$$

45. (a) Given: Line $\vec{r} = (1,-1,3) + t(2,-1,4)$, plane perpendicular to $\vec{n}_1 = (1,2,1)$
Step (1): Normal of required plane $\vec{n} \perp$ line direction and $\perp$ given plane:

$\vec{n} \cdot (2,-1,4) = 0, \vec{n} \cdot (1,2,1) = 0$
Solve: $\vec{n} = (-9,2,5)$
Step (2): Cartesian equation using point $(1,-1,3)$:
$-9(x-1) + 2(y+1) + 5(z-3) = 0 \Longrightarrow 9x - 2y - 5z + 4 = 0$
Step (3): Parametric form: directions (2, -1,4) and (1,2,1)
$\vec{r} = (1,-1,3) + s(2,-1,4) + t(1,2,1), s, t \in \mathbb{R}$

OR

(b) Given equation:

$6x^4 - 5x^3 - 38x^2 - 5x + 6 = 0, x = \frac{1}{3}$ is a root

Step (1): Factor out $3x - 1 \rightarrow$ divide polynomial:
$6x^4 - 5x^3 - 38x^2 - 5x + 6 = (3x-1)(2x^3 - x^2 - 13x - 6)$
Step (2): Factor cubic by trial: $x = 3$ is root $\rightarrow 2x^3 - x^2 - 13x - 6 = (x-3)(2x^2 + 5x + 2)$
Step (3): Factor quadratic: $2x^2 + 5x + 2 = (2x+1)(x+2)$
Step (4): All roots:
$x = \frac{1}{3}, 3, -\frac{1}{2}, -2$

46. (a) Step (1): Recall equivalence
By definition of implication:
$p \rightarrow X \equiv \neg p \vee X$
Here, $X = \neg q \vee r$.
So directly:
$p \rightarrow (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$
Step (2): Truth table (optional verification)

p	q	r	¬q	¬q v r	p → (¬q ∨ r)	¬pv (¬qvr)
T	T	T	F	T	T	T
T	T	F	F	F	F	F
T	F	T	T	T	T	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	T	F	F	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

All columns for $p \rightarrow (\neg q \vee r)$ and $\neg p \vee (\neg q \vee r)$ match

OR

(b) We are given:
- Principal: $P = 10{,}000$
- Rate: $r = 5\% = 0.05$ per annum
- Time: $t = 18$ months $= 1.5$ years
- Compounded continuously

Step (1): Formula for continuous compounding

$$A = Pe^{rt}$$

$A = 10000 \cdot e^{0.05 \cdot 1.5} = 10000 \cdot e^{0.075}$
Step (2): Approximate

$e^{0.075} \approx 1 + 0.075 + \frac{0.075^2}{2} = 1 + 0.075 + 0.0028125 \approx 1.0778125$

$A \approx 10000 \cdot 1.0778 \approx 10778$

47. (a) We are asked to find the maximum of:

$f(x)=\frac{\log x}{x}, x>0$

Step (1): Differentiate

$f'(x)=\frac{1\cdot x-\log x\cdot 1}{x^2}=\frac{1-\log x}{x^2}$

Set $f'(x)=0$:

$1-\log x=0 \Longrightarrow \log x=1 \Longrightarrow x=e$

Step (2): Maximum value

$f_{\max}=f(e)=\frac{\log e}{e}=\frac{1}{e}$

OR

(b) Change variables: $X=x/a, Y=y/b \rightarrow$ ellipse $X^2+Y^2=1$, line $X+Y=1$

Intersection points: (0,1), (1,0)

Area:

$A=ab\int_0^1\left(\sqrt{1-X^2}-(1-X)\right)dX=ab\left(\frac{\pi}{4}-\frac{1}{2}\right)=\frac{ab(\pi-2)}{4}$

$A=\frac{ab(\pi-2)}{4}$