PART - I

1. (a)
2. (d)
3. (a)
4. (d)
5. (d)
6. (a)
7. (d)
8. (b)
9. (d)
10. (a)
11. (c)
12. (d)
13. (c)
14. (a)
15. (d)
16. (b)
17. (a)
18. (c)
19. (d)
20. (c)

PART - II

21. Powers of i repeat every 4: $i, -1, -i, 1, \ldots$

$$\sum_{n=1}^{12} i^n = \text{ sum of 3 full cycles of 4 terms each } = 0$$

22. Original quadratic: $2x^2 - 7x + 13 = 0 \rightarrow \alpha + \beta = 7/2, \alpha\beta = 13/2$

- Sum of new roots: $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = (7/2)^2 - 2(13/2) = -3/4$
- Product of new roots: $\alpha^2\beta^2 = (\alpha\beta)^2 = (13/2)^2 = 169/4$

New quadratic: $x^2 - (\text{ sum })x + (\text{ product }) = x^2 + 3/4x + 169/4 = 0$

Multiply by 4: $4x^2 + 3x + 169 = 0$

23. $f(x) = x^2 + 3x \Longrightarrow df = f'(x)dx = (2x+3)dx$

At $x = 3, dx = 0.02$:

$df = (2 \cdot 3 + 3)(0.02) = 9 \cdot 0.02 = 0.18$

24. - Family of lines through the origin: $y = mx$, where m is an arbitrary constant.

- Differentiate w.r.t x : -

$$\frac{dy}{dx} = m$$

- But $m = \frac{y}{x}$, so

$$\frac{dy}{dx} = \frac{y}{x} \Longrightarrow x\frac{dy}{dx} - y = 0$$

Differential equation: -

$$x\frac{dy}{dx} - y = 0$$

25. Here, $f(x) = 2(x-1)$ on $[1,2]$:

$$E[X] = \int_1^2 x \cdot 2(x-1)dx = 2\int_1^2 x(x-1)dx$$

$$x(x-1) = x^2 - x$$

$$E[X] = 2\int_1^2 (x^2 - x)dx = 2\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2$$

Compute:

$\left[\frac{8}{3}-2\right]-\left[\frac{1}{3}-\frac{1}{2}\right]=\left(\frac{8}{3}-\frac{6}{3}\right)-\left(\frac{1}{3}-\frac{1}{2}\right)=\frac{2}{3}-\left(-\frac{1}{6}\right)=\frac{2}{3}+\frac{1}{6}=\frac{5}{6}$

Multiply by 2:

$E[X]=2\cdot\frac{5}{6}=\frac{5}{3}$

26. General equation of a circle with center (h,k) and radius r :

$(x-h)^2+(y-k)^2=r^2$

Here: $h=-3,k=-4,r=3$

$(x+3)^2+(y+4)^2=9$

27. Matrix A is 3×2. Maximum rank $=2$.

Check if columns are linearly independent:

Column 1: $\begin{bmatrix}-1\\4\\3\end{bmatrix}$

Suppose $c_1\cdot\text{col1}+c_2\cdot\text{col2}=0$:

$c_1(-1)+c_2(3)=0\Longrightarrow -c_1+3c_2=0\Longrightarrow c_1=3c_2$

$c_1(4)+c_2(-7)=0\Longrightarrow 4c_1-7c_2=0\Longrightarrow 4(3c_2)-7c_2=5c_2=0\Longrightarrow c_2=0\Longrightarrow c_1=0$

Columns are independent $\rightarrow$ rank $=2$

28. Step by step:

(i) $I_{10}=\frac{9}{10}I_8$

(ii) $I_8=\frac{7}{8}I_6\rightarrow I_{10}=\frac{9}{10}\cdot\frac{7}{8}I_6$

(iii) $I_6=\frac{5}{6}I_4\rightarrow I_{10}=\frac{9}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}I_4$

(iv) $I_4=\frac{3}{4}I_2\rightarrow I_{10}=\frac{9}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}I_2$

(v) $I_2=\frac{1}{2}I_0=\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4}$

Multiply fractions:

$I_{10}=\frac{9\cdot 7\cdot 5\cdot 3}{10\cdot 8\cdot 6\cdot 4}\cdot\frac{\pi}{4}=\frac{945}{1920}\cdot\frac{\pi}{4}=\frac{945\pi}{7680}=\frac{63\pi}{512}$

29. Factor numerator and denominator:

$\frac{x^2-3x+2}{x^2-4x+3}=\frac{(x-1)(x-2)}{(x-1)(x-3)}=\frac{x-2}{x-3},x\neq 1$

Substitute $x=1$:

$\frac{1-2}{1-3}=\frac{-1}{-2}=\frac{1}{2}$

30. Step (1): Compute $\vec{b}\times\vec{c}$

$\vec{b}=(1,-1,0),\vec{c}=(3,-1,6)$

$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{\imath}&\hat{\jmath}&\hat{k}\\1&-1&0\\3&-1&6\end{vmatrix}=\hat{\imath}((-1)(6)-0(-1))-\hat{\jmath}(1\cdot 6-0\cdot 3)+\hat{k}(1\cdot(-1)-(-1)\cdot 3)$

$=\hat{\imath}(-6)-\hat{\jmath}(6)+\hat{k}(-1+3)=-6\hat{\imath}-6\hat{\jmath}+2\hat{k}$

Step (2): Compute $\vec{a}\cdot(\vec{b}\times\vec{c})$

$\vec{a}\cdot(\vec{b}\times\vec{c})=(2)(-6)+(-1)(-6)+(3)(2)=-12+6+6=0$

Step (3): Conclusion

$[\vec{a},\vec{b},\vec{c}]=0\Longrightarrow\vec{a},\vec{b},\vec{c}$ are coplanar.

PART - III

31. Let

$y=\cot^{-1}\frac{1}{\sqrt{x^2-1}}\Longrightarrow\cot y=\frac{1}{\sqrt{x^2-1}}$

Take reciprocal:

$\tan y=\sqrt{x^2-1}\Longrightarrow\sec^2 y=1+\tan^2 y=1+(x^2-1)=x^2$

$\sec y=x$ (since $|x|>1$)

$\Longrightarrow y=\sec^{-1}x$

32. Given parabola: $x^2+6x+4y+5=0$ and point $(1,-3)$
(i) Slope of tangent:
$$\frac{dy}{dx}=-\frac{2x+6}{4}=-\frac{x+3}{2}\Longrightarrow m_{\text{tangent}}=-2$$
(ii) Tangent equation:
$y+3=-2(x-1)\Longrightarrow y=-2x-1$
(iii) Normal equation:
$$m_{\text{normal}}=\frac{1}{2},y+3=\frac{1}{2}(x-1)\Longrightarrow x-2y-7=0$$

33. Given the scalar triple product
$[\vec{a}-\vec{b},\vec{b}-\vec{c},\vec{c}-\vec{a}]$.
Note that
$(\vec{a}-\vec{b})+(\vec{b}-\vec{c})+(\vec{c}-\vec{a})=\vec{0}$.
Hence the three vectors are linearly dependent.
The scalar triple product of linearly dependent vectors is zero.
$\boxed{[\vec{a}-\vec{b},\vec{b}-\vec{c},\vec{c}-\vec{a}]=0.}$

34. Given
$$u(x,y)=\frac{x^2+y^2}{\sqrt{x+y}}=(x^2+y^2)(x+y)^{-1/2}$$
This is a homogeneous function.
- Degree of $x^2+y^2=2$
- Degree of $(x+y)^{-1/2}=-\frac{1}{2}$

So, degree of u :
$$n=2-\frac{1}{2}=\frac{3}{2}$$
By Euler's theorem on homogeneous functions,
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$
$$\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{3}{2}u}$$

35. Let the two positive numbers be x and $12-x$.
Their product:
$P=x(12-x)=12x-x^2$
Differentiate:
$$\frac{dP}{dx}=12-2x$$
Set $\frac{dP}{dx}=0$:
$12-2x=0\Rightarrow x=6$
So, the other number is $12-6=6$.
$\boxed{\text{The two numbers are 6 and 6.}}$

36. $I=\int_{\pi/8}^{3\pi/8}\frac{dx}{1+\sqrt{\tan x}}$
Use the property:
$$\int_a^b f(x)dx=\int_a^b f\left(\frac{\pi}{2}-x\right)dx$$
So,
$$I=\int_{\pi/8}^{3\pi/8}\frac{dx}{1+\sqrt{\cot x}}$$
Add the two forms:
$$2I=\int_{\pi/8}^{3\pi/8}\left(\frac{1}{1+\sqrt{\tan x}}+\frac{1}{1+\sqrt{\cot x}}\right)dx$$
Note:

$$\frac{1}{1+\sqrt{\tan x}}+\frac{1}{1+\sqrt{\cot x}}=1$$

Hence,

$$2I=\int_{\pi/8}^{3\pi/8} 1dx=\frac{3\pi}{8}-\frac{\pi}{8}=\frac{\pi}{4}$$

$$\boxed{I=\frac{\pi}{8}}$$

37. $\left(\frac{1+i}{1-i}\right)=\frac{(1+i)^2}{(1-i)(1+i)}=\frac{1+2i+i^2}{1-i^2}=\frac{2i}{2}=i$

Similarly,

$$\frac{1-i}{1+i}=-i$$

Now,

$$\left(\frac{1+i}{1-i}\right)^3-\left(\frac{1-i}{1+i}\right)^3=i^3-(-i)^3$$

$$=(-i)-i=-2i$$

38. $(1+x^2)\frac{dy}{dx}=1+y^2$

Separate variables:

$$\frac{dy}{1+y^2}=\frac{dx}{1+x^2}$$

Integrate both sides:

$$\tan^{-1}y=\tan^{-1}x+C$$

Hence,

$$\boxed{\tan^{-1}y-\tan^{-1}x=C}$$

39. Let X be the number of heads when three fair coins are tossed.

Possible values: $x=0,1,2,3$

$$P(X=0)=\frac{1}{8}, P(X=1)=\frac{3}{8}, P(X=2)=\frac{3}{8}, P(X=3)=\frac{1}{8}$$

This is the probability mass function (PMF).

40. Given

$$A=\begin{bmatrix}2 & -1 & 3\\ -5 & 3 & 1\\ -3 & 2 & 3\end{bmatrix}$$

First find $|A|$:

$$|A|=2(3\cdot 3-1\cdot 2)-(-1)(-5\cdot 3-1\cdot(-3))+3(-5\cdot 2-3\cdot(-3))$$

$$=2(7)-(-1)(-12)+3(-1)=14-12-3=-1$$

For a 3×3 matrix,

$$\text{adj}(\text{adj}A)=|A|^1A$$

Taking determinant:

$$|\text{adj}(\text{adj}A)|=|A|^4$$

$$=(-1)^4=1$$

$$|\text{adj}(\text{adj}A)|=1$$

PART - IV

41. (a) Curves:

$y=x^2$ and $y=(x-3)^2$

Point of intersection:

$$x^2=(x-3)^2\Rightarrow 6x=9\Rightarrow x=\frac{3}{2}$$

Slopes at intersection:

$$\frac{dy}{dx}=2x\Rightarrow m_1=2\left(\frac{3}{2}\right)=3$$

$$\frac{dy}{dx}=2(x-3)\Rightarrow m_2=2\left(\frac{3}{2}-3\right)=-3$$

Angle between curves:

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{3-(-3)}{1-9}\right| = \frac{6}{8} = \frac{3}{4}$$

$$\theta = \tan^{-1}\left(\frac{3}{4}\right)$$

OR

(b) Given

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$

Use:

$$\tan^{-1} a + \tan^{-1} b = \tan^{-1}\left(\frac{a+b}{1-ab}\right) \text{ (when valid)}$$

Let

$$a = \frac{x-1}{x-2}, b = \frac{x+1}{x+2}$$

Compute:

$$a + b = \frac{(x-1)(x+2) + (x+1)(x-2)}{(x-2)(x+2)} = \frac{2x^2 - 4}{x^2 - 4}$$

$$ab = \frac{x^2 - 1}{x^2 - 4}$$

$$\frac{a+b}{1-ab} = \frac{\frac{2(x^2-2)}{x^3-4}}{\frac{-3}{x^2-4}} = -\frac{2(x^2-2)}{3}$$

So,

$$\tan^{-1}\left(-\frac{2(x^2-2)}{3}\right) = \frac{\pi}{4}$$

$$-\frac{2(x^2-2)}{3} = 1 \Rightarrow x^2 = \frac{1}{2}$$

$$\boxed{x = \pm\frac{1}{\sqrt{2}}}$$

42. (a) One throw outcome:

$$P(1) = \frac{1}{6}, P(3) = \frac{1}{3}, P(5) = \frac{1}{2}$$

Let X = total score in two throws.

(i) PMF

Possible values: 2, 4, 6, 8, 10

$$P(X = 2) = \frac{1}{30}$$

$$P(X = 4) = \frac{1}{9}$$

$$P(X = 6) = \frac{5}{18}$$

$$P(X = 8) = \frac{1}{3}$$

$$P(X = 10) = \frac{1}{4}$$

(ii) CDF $F(x) = P(X \le x)$

$$F(x)=\begin{cases}0, & x<2\\ \frac{1}{36}, & 2\le x<4\\ \frac{5}{36}, & 4\le x<6\\ \frac{5}{12}, & 6\le x<8\\ \frac{3}{4}, & 8\le x<10\\ 1, & x\ge 10\end{cases}$$

(iii)

$$P(4\le X<10)=P(4)+P(6)+P(8)=\frac{13}{18}$$

OR

(b)

$z=x+iy$

Then

$2z+1=(2x+1)+i(2y), iz+1=(1-y)+ix$

$$\frac{2z+1}{iz+1}=\frac{(2x+1)+i(2y)}{(1-y)+ix}$$

Multiply numerator and denominator by the conjugate $(1-y)-ix$:

$$=\frac{[(2x+1)+i(2y)]((1-y)-ix)}{(1-y)^2+x^2}$$

Imaginary part of numerator:

$=2y(1-y)-x(2x+1)$

Given $\mathrm{Im}\left(\frac{2z+1}{iz+1}\right)=0$,

$2y(1-y)-x(2x+1)=0$

$2y-2y^2-2x^2-x=0$

$2x^2+2y^2+x-2y=0$

43. (a) Given

Height of cone $H=12$ m

Radius at top $R=5$ m

Rate of inflow $\frac{dV}{dt}=10\ \mathrm{m}^3/\mathrm{min}$

Let

Depth of water $=h$ m

Radius of water surface $=r$ m

Step 1: Relation between r and h

By similarity of triangles:

$$\frac{r}{h}=\frac{5}{12}\Rightarrow r=\frac{5h}{12}$$

Step 2: Volume of water

$$V=\frac{1}{3}\pi r^2h$$

Substitute $r=\frac{5h}{12}$:

$$V=\frac{1}{3}\pi\left(\frac{5h}{12}\right)^2h=\frac{25\pi}{432}h^3$$

Step 3: Differentiate w.r.t. time

$$\frac{dV}{dt}=\frac{25\pi}{432}\cdot 3h^2\frac{dh}{dt}=\frac{25\pi}{144}h^2\frac{dh}{dt}$$

Step 4: Put given values

When $h=8$ m and $\frac{dV}{dt}=10$:

$$10=\frac{25\pi}{144}\times 64\times\frac{dh}{dt}$$

$$10 = \frac{100\pi}{9}\frac{dh}{dt}$$

Step 5: Solve

$$\frac{dh}{dt} = \frac{9}{10\pi}$$

OR

(b) Vector proof of

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Let $\hat{\imath}$ and $\hat{\jmath}$ be unit vectors along the x - and y-axes.

Define two unit vectors:

$\vec{a} = \cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}$

$\vec{b} = \cos\beta\hat{\imath} + \sin\beta\hat{\jmath}$

Step 1: Angle between vectors

The angle between $\vec{a}$ and $\vec{b}$ is $(\alpha - \beta)$.

Step 2: Use cross product

Magnitude of cross product:

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin(\alpha - \beta)$

Since both are unit vectors:

$|\vec{a} \times \vec{b}| = \sin(\alpha - \beta)$

Step 3: Compute $\vec{a} \times \vec{b}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \end{vmatrix}$$

$= \hat{k}(\cos\alpha\sin\beta - \sin\alpha\cos\beta)$

So,

$|\vec{a} \times \vec{b}| = |\sin\alpha\cos\beta - \cos\alpha\sin\beta|$

Step 4: Compare both results

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

44. **(a)** Take the end of the pipe as origin $O(0,0)$.

- x-axis: horizontal (outward)
- y-axis: vertical downward

Since the vertex is at the pipe end, the path is a parabola:

$y = ax^2$

Given point on the path:

2.5 m below the pipe and 3 m outward $\Rightarrow (x, y) = (3,2.5)$

$$2.5 = a(3)^2 \Rightarrow a = \frac{2.5}{9}$$

So, equation:

$$y = \frac{2.5}{9}x^2$$

Water strikes the ground 7.5 m below the pipe $\Rightarrow y = 7.5$

$$7.5 = \frac{2.5}{9}x^2 \Rightarrow x^2 = 27 \Rightarrow x = 3\sqrt{3}$$

Water strikes the ground $3\sqrt{3}$ m beyond the vertical line.

OR

(b) $\frac{dy}{dx} + \frac{1}{x}y = \sin x$

This is a linear differential equation.

Integrating factor:

$\text{IF} = e^{\int \frac{1}{x}dx} = x$

Multiply throughout by x :

$$x\frac{dy}{dx} + y = x\sin x$$

$\frac{d}{dx}(xy) = x\sin x$

Integrate:

$xy = \int x\sin x dx$

Using integration by parts:

$\int x\sin x dx = -x\cos x + \sin x + C$

Hence,

$\boxed{y = \frac{\sin x - x\cos x + C}{x}}$

45. (a) Let $N(t)$ be the number of bacteria at time t.

Given rate $\propto N$:

$\frac{dN}{dt} = kN$

Solution:

$N = N_0 e^{kt}$

Given the number triples in 5 hours:

$3N_0 = N_0 e^{5k} \Rightarrow e^{5k} = 3$

After 10 hours:

$N(10) = N_0 e^{10k} = \left(e^{5k}\right)^2 N_0 = 3^2 N_0 = 9N_0$

$\boxed{\text{After 10 hours, the number of bacteria is 9 times the initial number.}}$

OR

(b) Direction vector:

$\vec{d}_2 = \langle 3,2,1 \rangle$

A plane containing L_1 and parallel to L_2 contains both direction vectors $\vec{d}_1, \vec{d}_2$.

Point on the plane (from L_1):

$\vec{r}_0 = \langle 2,2,1 \rangle$

Vector equation of the plane

$\vec{r} = \langle 2,2,1 \rangle + \lambda\langle 2,3,3 \rangle + \mu\langle 3,2,1 \rangle$

Cartesian equation

Normal vector:

$$\vec{n} = \vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 3 \\ 3 & 2 & 1 \end{vmatrix} = \langle -3,7,-5 \rangle$$

Plane equation:

$-3(x-2) + 7(y-2) - 5(z-1) = 0$

$\boxed{3x - 7y + 5z + 3 = 0}$

46. (a) Ellipse:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Write in parametric form:

$x = a\cos\theta, y = b\sin\theta$

Area enclosed by ellipse:

$A = \pi ab$

OR

(b) Given parabola:

$y^2 - 4y - 8x + 12 = 0$

Complete the square in y :

$(y^2 - 4y + 4) - 4 - 8x + 12 = 0$

$(y-2)^2 = 8(x-1)$

Compare with standard form $(y-k)^2 = 4a(x-h)$:

- $h = 1, k = 2$
- $4a = 8 \Rightarrow a = 2$

Vertex

(1,2)
Focus
$(h+a,k)=(1+2,2)=(3,2)$
Directrix
$x=h-a=1-2=-1$

47. (a) We know,
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
Now,
$p \rightarrow q \equiv (\neg p) \vee q$
$q \rightarrow p \equiv (\neg q) \vee p$
Substitute:
$p \leftrightarrow q \equiv ((\neg p) \vee q) \wedge ((\neg q) \vee p)$
$\boxed{p \leftrightarrow q \equiv ((\sim p) \vee q) \wedge ((\sim q) \vee p)}$

OR

(b) Let
$u=\frac{1}{x}, v=\frac{1}{y}, w=\frac{1}{z}$
Then the system becomes:
$$\begin{cases} 3u-4v-2w=1 \\ u+2v+w=2 \\ 2u-5v-4w=-1 \end{cases}$$
Determinant:
$$\Delta=\begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & -5 & -4 \end{vmatrix}=-1$$
$$\Delta_u=\begin{vmatrix} 1 & -4 & -2 \\ 2 & 2 & 1 \\ -1 & -5 & -4 \end{vmatrix}=-1 \Rightarrow u=\frac{\Delta_u}{\Delta}=1$$
$$\Delta_v=\begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & -4 \end{vmatrix}=2 \Rightarrow v=\frac{2}{-1}=-2$$
$$\Delta_w=\begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 2 \\ 2 & -5 & -1 \end{vmatrix}=-1 \Rightarrow w=\frac{-1}{-1}=1$$
Thus,
$\frac{1}{x}=1, \frac{1}{y}=-2, \frac{1}{z}=1$
$x=1, y=-\frac{1}{2}, z=1$